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# A FAST COMPUTATIONAL PROCEDURE TO SOLVE THE MULTI-ITEM SINGLE MACHINE LOT SCHEDULING OPTIMIZATION PROBLEM

# UNE PROCÉDURE NUMÉRIQUE POUR L'OPTIMISATION D'UN SYSTÈME DES MACHINES MULTIPRODUITS

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**Abstract**

We present in this paper some especial procedures for the numerical solution of the optimal schedule problem of a multi-item single machine. A method of discretization and a computational procedure are described which allows us to calculate the solution in a short time and with a precision of order $k$, being $k$ the size of the discretization. The principal feature of this method is the fact that the nodes of the triangulation mesh are joined by segments of trajectories of the original system. This feature allows us to obtain the $k$-order precision which, in general, is impossible to obtain by usual methods. We also develop a highly efficient algorithm that converges in a finite number of steps.

Résumé

On présente ici quelques procédés spéciaux pour la solution numérique du problème d'optimisation d'un système de production qui est composé d'une machine multi-produit. On introduit une méthode de discretisation et un procédé computationnel, qui permetent d'obtenir la solution en employant des petits temps de calcul et avec une précision d'ordre $k$, étant $k$ la mesure de la discretisation. La caractéristique principale de cette méthode est le fait que les nodes de la triangulation sont unis par des segments des trajectoires du système originel. Cette caractéristique permet d'obtenir la précision d'ordre $k$, laquelle en général, est impossible d'obtenir par les méthodes habituelles. On a développé aussi un algorithme computationnel très efficient qui converge dans un nombre fini des pas.

**Keywords:** *scheduling problems, quasi-variational inequalities, Bellman equation, numerical solution.*

| | | |
|---|---|---|
| **AMS Classification:** | Primary: | 90B30, 90B35, 90B05, 90B10 |
| | Secondary: | 90C39, 49B10, 65K10 |

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# 1 Introduction

We present a fast numerical method to optimize the production schedule of a multi-item single machine. Our objective is to find an optimal production schedule that minimizes a functional $J$, which involves instantaneous and switching costs and has the following form

$$J(\alpha(\cdot)) = \sum_{i=1}^{\infty} \left( \int_{\theta_{i-1}}^{\theta_i} f(y(s), \alpha(s))\, e^{-\lambda s}\, ds + q(d_{i-1}, d_i)\, e^{-\lambda\theta_i} \right). \tag{1}$$

For each $d \in D = \{0, 1, ..., m\}$ and $x \in Q = \prod_{i=1}^{m} [0, M_i]$, we define the value function

$$U_d(x) \equiv \inf \left\{ J(\alpha(\cdot)) \ : \ \alpha(\cdot) \in A_x^d \right\}, \tag{2}$$

which is the minimum cost of operation starting from the initial state $x$ and the initial state of production $d$. We denote $A_x^d$ the set of admissible policies starting from those initial conditions.

Our objective is to find, for each $x \in Q$ and $d \in D$, an optimal schedule $\bar{\alpha}(\cdot)$, i.e.

$$\bar{\alpha}(\cdot) \in A_x^d, \text{ such that } J(\bar{\alpha}(\cdot)) = U_d(x). \tag{3}$$

In [10], we can see how an optimal feedback policy can be found in terms of the optimal cost function $U_d$ . This work presents a numerical method which allows us to calculate $U_d$ with a fast procedure.
Related results can be seen in [1], [2], [12], [13].

The more important properties of our method are the followings

- The discrete approximations have precision of order $k$.
- The iterative algorithm converges in a finite number of steps.
- Each step involves the solution of a linear system, which can be solved in a very efficient ad-hoc form.

# 2 Description of the control problem

## 2.1 Description of the production system

At any time the machine is either idle or producing any one of $m$ different items. We will denote by

- $d = 0$ the idle state of the machine
- $d = 1, ..., m$, when it is producing item $d$.

For each item $d = 1, ..., m$; we define the problem data as follow

- $r_d$ the demand by unit time of item $d$
- $p_d$ the production rate by unit time at the machine setting $d$
- $M_d$ the inventory capacity constraint of item $d$
- $q(d, \tilde{d})$ the switching cost of the machine from state $d$ to $\tilde{d}$
- $f(x, d)$ the instantaneous inventory-holding/production cost
- $\lambda$ the discount rate.

We will always assume the non zero loop-cost condition: $\exists q_0 > 0$ such that for any closed loop $d_0, d_1, ..., d_p, d_{p+1}$ , with $d_0 = d_{p+1}$ , $p \leq m$, we have

$$\sum_{i=0}^{p} q(d_i, d_{i-1}) \geq q_0 \tag{4}$$

and we suppose that the following conditions are verified

$$q(d, \tilde{d}) \geq 0 \quad \forall \tilde{d} \neq d, \qquad q(d, d) = 0 \qquad \forall d \in D, \tag{5}$$

$$q(d, \bar{d}) \leq q(d, \tilde{d}) + q(\tilde{d}, \bar{d}), \qquad \forall d \neq \tilde{d} \neq \bar{d}. \tag{6}$$

In addition, we assume that the switching time is small enough to be disregarded and that the following condition, under which a feasible schedule exist, holds

$$\sum_{d=1}^{m} \frac{r_d}{p_d} \leq 1. \tag{7}$$

In fact, we will always assume only the sign "$<$" holding in (7), because condition $\sum_{d=1}^{m} \frac{r_d}{p_d} = 1$, forbids the machine to be in the idle state except for a total time $\tau = \sum_{d=1}^{m} \frac{x_d}{p_d}$, and this is not a reasonable condition for a problem with infinite horizon.

## 2.2 The set $Q$ of admissible states

Let $y_d(t)$ be the inventory level of item $d$ at time $t$, starting at $y_d(0) = x_d$. Therefore, for the global state "$y$" of the system, we have

$$
\begin{gathered}
y(t) = (y_1(t), ..., y_m(t)) \\
(y_1(0), ..., y_m(0)) = (x_1, ..., x_m).
\end{gathered}
\tag{8}
$$

As neither backlogging nor production over the capacity constraints are allowed for the inventory state $y_d$, the following restriction holds

$$
0 \leq y_d \leq M_d, \qquad \forall d = 1, ..., m. \tag{9}
$$

Let us divide the $x_i$ values into three categories

$$
\left|\begin{array}{l}
x_i = 0, \\
0 < x_i < M_i, \\
x_i = M_i.
\end{array}\right. \tag{10}
$$

A point $x$ is classified using an m-tuple of digits $a(x) = (a_1, ..., a_m)$, where

$$
\left|\begin{array}{lcl}
x_i = 0 & \Rightarrow & a_i = 0 \\
0 < x_i < M_i & \Rightarrow & a_i = 1 \\
x_i = M_i. & \Rightarrow & a_i = 2.
\end{array}\right. \tag{11}
$$

Let us define

$$
\Gamma(a_1, ..., a_m) \equiv \{x \ : \ a(x) = (a_1, ..., a_m)\}.
$$

The set $Q$ of admissible states comprises only the set of points with at most one zero component; because if we start from other points that do not verify this condition, we cannot avoid the shortage of at least one item, i.e.

$$
Q = \bigcup_a \{\Gamma(a_1, ..., a_i, ..., a_m) \ : \ \text{at most one component } a_i = 0\}. \tag{12}
$$

Let us denote with $\partial Q^+$ the points of $Q$ that are not admissible, i.e.

$$
\partial Q^+ = \bigcup_a \{\Gamma(a_1, ..., a_i, ..., a_m) \ : \ \text{at least two } a_i = 0\}.
$$

If we denote with $\Omega$ the interior of Q, we have

$$
\Omega \equiv \{x \ : \ 0 < x_i < M_i, \ i = 1, ..., m\} = \Gamma(1, ..., 1). \tag{13}
$$

## 2.3 The evolution of the system

For any step function $\alpha : [0, \infty) \to D$ from the definition of $r_d$, $p_d$, the following equation of evolution holds

$$\frac{dy}{dt} = g(\alpha(t)), \tag{14}$$

where

$$g(\alpha) = (g_1(\alpha), ..., g_m(\alpha)), \tag{15}$$

being

$$g_d(\alpha) = \begin{vmatrix} -r_d & \text{if } \alpha \neq d, \\ \\ p_d - r_d & \text{if } \alpha = d. \end{vmatrix}$$

**Remark 2.1** Since $g$ is piece-wise constant, the equation (14) has global solution for any control policy. At the same time we always suppose that the function $f$ is uniformly Lipschitz-continuous in $Q$, $\forall d \in D$.

## 2.4 The set $A_x^d$ of admissible controls

An admissible schedule is characterized by a sequence of pairs $\{\theta_i, d_i\}$, where $\theta_i$ is a generic switching time,

$$0 \leq \theta_0 \leq \theta_1 < ... < \theta_i < \theta_{i+1} < ... \tag{16}$$

and $d_i \in D$; $d_i \neq d_{i+1}$; $i = 0, 1, ...$ is the state of production in $(\theta_i, \theta_{i+1}]$ .
For each $x \in Q$, $d \in D$, we denote $A_x^d$ the set of all admissible schedules with initial state $x$ and initial machine setting $d$

$$A_x^d = \left\{\alpha(\cdot) = \{\theta_i, d_i\}_{i=0}^{\infty} \ : \ d_0 = d, \ y(t) \in Q \ \forall t \in \Re^+\right\}. \tag{17}$$

In other words, we will consider sequences $\{\theta_i, d_i\}$ such that the associated trajectories remain in $Q$, $\forall t \geq 0$.

## 2.5 The optimal cost function $U$

We consider the cost function defined by (1) and the optimal cost function $U(x)$, $U$ is a vector with components $U_d$, defined in (2). By virtue of the properties of the dynamical system studied, some properties of regularity for functions $U_d$ hold; in particular, they are locally Lipschitz continuous. On each compact subset of $\Omega$, the Lipschitz coefficient is independent of the parameter $\lambda$. This property is crucial for the estimation of the rate of convergence of the numerical solutions and also for the study of the optimization problem with the time average criterion.

# 3 Properties of the optimal cost function

Hypotheses (5), (4), (14), together with feasibility condition (7), allow the proof of strong regularity properties. These properties are detailed in section 3.1. The proof of these properties can be seen in [10].

**Preliminary remark:** The feasibility condition (7) implies a property of controllability that plays a key role to prove that the optimal cost function is bounded and locally Lipschitz-continuous. These properties are described in the following sections.

## 3.1 Property of local boundness

**Property:** To find the policy that realizes the minimum value $U_d(x)$, it is enough to restrict the election to those policies that verify

$$J(\alpha(\cdot)) \leq C_J\left(1 + \frac{1}{\lambda} + \left(\log(d(x, \partial Q^+))\right)^-\right). \tag{18}$$

In this form we can obtain an estimation for the density of switching points. This estimation only depends on the initial point of the trajectory.

**Theorem 3.1** *Let*

$$T = \max\left(\frac{2}{\zeta}\sum_{i=1}^{m}\frac{M_i}{p_i},\ 2\max\left(\frac{M_i}{r_i}\right)\right), \tag{19}$$

*where*

$$\zeta = 1 - \sum_{i=1}^{m}\frac{r_i}{p_i}.$$

*Let $x \in Q$, $d \in D$ and $\bar{\alpha}$ be an optimal policy for the initial conditions $(x, d)$. If we denote by $\nu_x$ the number of switchings in [0,T] of $\bar{\alpha}$, $\nu_x$ verifies*

$$\nu_x \leq \overline{C}\left(1 + \left(\log(d(x, \partial Q^+))\right)^-\right). \tag{20}$$

**Theorem 3.2** *$U$ is locally bounded, i.e. $\exists C_U > 0$ such that*

$$0 \leq U_d(x) \leq C_U\left(1 + \frac{1}{\lambda} + \left(\log(d(x, \partial Q^+))\right)^-\right), \quad \forall x \in Q, \forall d \in D. \tag{21}$$

*$U$ is unbounded in neighborhoods of $\partial Q^+$ and $\exists c > 0$ that satisfies the following inequality*

$$U_d(x) \geq c\left(\log(d(x, \partial Q^+))\right)^-, \quad \forall x \in Q, \forall d \in D.$$

## 3.2 Property of local Lipschitz-continuity

**Theorem 3.3** *$U$ is locally Lipschitz continuous, i.e. $\exists$ a non-increasing function $L(\cdot)$ such that $\forall x \in Q$, $\forall x' \in Q$, $\forall d \in D$*

$$|U_d(x) - U_d(x')| \leq L(\eta)\, \|x - x'\|\,, \tag{22}$$

*where*

$$\eta = \min\left(d(x, \partial Q^+),\ d(x', \partial Q^+)\right).$$

**Remark 3.1** From Theorem 3.3 it results that $U$ is uniformly Lipschitz continuous outside any neighborhood of $\partial Q^+$.

## 3.3 Construction of an optimal feedback policy

We define

$$S^d(U)(x) = \min_{\tilde{d} \neq d} \left(q(d, \tilde{d}) + U_{\tilde{d}}(x)\right) \qquad x \in Q,\ d \in D. \tag{23}$$

The optimal cost function and the operator $S^d$ can be used to define an optimal control policy in the following way (see [10]).

**Theorem 3.4** *For any state $(x, d)$, there exists at least an optimal control policy $\bar{\alpha}(\cdot)$, i.e.*

$$U_d(x) = J(\bar{\alpha}(\cdot)). \tag{24}$$

*An optimal feedback control policy $\bar{\alpha} = \{\theta_i, d_i\}_{i=0}^{\infty} \in A_x^d$, can be obtained in terms of $U_d$ in the following way:*

$$\theta_0 = 0,\ d_0 = d$$

*and recursively*

$$\begin{gathered} \theta_i = \min\left\{t \geq \theta_{i-1}\ :\ U_{d_{i-1}}(y(t)) = \left(S^{d_{i-1}}(U)\right)(y(t))\right\}, \\ d_i \in \left\{d \in D\ :\ d \neq d_{i-1}\,,\ \left(S^{d_{i-1}}(U)\right)(y(\theta_i)) = U_d(y(\theta_i)) + q(d_{i-1}\,, d)\right\}. \end{gathered} \tag{25}$$

**Theorem 3.5** *Let $T$ be given by (19), then there exists $L_m > 0$ such that $\forall x \in Q, \forall d \in D, \forall i \in D$ it is verified*

$$m\left(\{t \in \{0, T]\ :\ \bar{\alpha}_{x,d}(t) = i\}\right) \geq L_m\,. \tag{26}$$

# 4 Dynamic programming solution

The Dynamic Programming Principle allows us to obtain the Hamilton-Jacobi-Bellman (H-J-B) equations associated to this problem, either in integral form or in differential form, and its boundary conditions. The proof uses classical techniques and it will not be included here for the sake of briefness [6].

## 4.1 The H-J-B equation in integral form

**Theorem 4.1** *For each $d \in D$ and $x \in \Omega$, the following conditions are verified*

$$U_d(x) \leq S^d(U)(x) \qquad x \in \Omega,\ d \in D, \tag{27}$$

$$U_d(x) \leq \int_0^t f(y(s), d)\ e^{-\lambda s}\ ds + U_d(y(t))\ e^{-\lambda t} \qquad \forall t > 0\ /\ y(t) = x + tg(d) \in \Omega\,. \tag{28}$$

*If furthermore, for some point $x \in \Omega$, a strict inequality holds in (27), then there exists $t_x > 0$ such that*

$$U_d(x) = \int_0^t f(y(s), d)\ e^{-\lambda s}\ ds + U_d(y(t))\ e^{-\lambda t} \qquad \forall 0 < t \leq t_x\,. \tag{29}$$

## 4.2 Boundary conditions for the H-J-B equation

Since the system is constrained to the set $Q$, conditions involving the values of function $U$ appear on the boundary of $Q$ (see [10], [12], [13]). This conditions depend only on the values of $U$ and they do not involve its derivatives - as occurs in problems with continuous controls, where constraints concerning the values of the Hamiltonian appear (see [4], [19]).

**Definition 4.1**

$$\begin{aligned} \gamma_d^+ &= \bigcup_a \{\Gamma(a_1, ..., a_d, ..., a_m)\ :\ a_d = 2\} \bigcap Q\,, \\ \gamma_d^- &= \bigcup_a \{\Gamma(a_1, ..., a_d, ..., a_m)\ :\ a_d = 0\} \bigcap Q \end{aligned} \tag{30}$$

and

$$\partial Q_e = \bigcup_{i=1}^{m} \left(\gamma_i^+ \cup \gamma_i^-\right). \tag{31}$$

**Theorem 4.2** *In $\partial Q_e$ the following boundary conditions are verified*

$$U_{\tilde{d}}(x) = \left(S^{\tilde{d}} U\right)(x) \qquad \forall x \in \gamma_d^-\,,\ \forall \tilde{d} \neq d, \tag{32}$$

$$U_d(x) = \left(S^d U\right)(x) \qquad d \neq 0,\ \forall x \in \gamma_d^+\,. \tag{33}$$

## 4.3 The H-J-B equation in differential form

The optimal cost function $U$ is uniformly Lipschitz continuous on compact subsets of $\Omega$ and then it is differentiable a.e. [7]. In consequence, we can associate to the optimal cost function the following H-J-B inequality in terms of its derivatives.

**Theorem 4.3** *For each $d \in D$, the following relations are verified*

$$U_d(x) \leq S^d(U)(x) \qquad \forall x \in Q, \tag{34}$$

$$\lambda U_d(x) - \frac{\partial U_d(x)}{\partial x}\, g(d) - f(x,d) \leq 0 \qquad a.e.\ x \in Q, \tag{35}$$

$$\left(U_d(x) - S^d(U)(x)\right)\left(\lambda U_d(x) - \frac{\partial U_d(x)}{\partial x}\, g(d) - f(x,d)\right) = 0 \qquad a.e.\ x \in Q, \tag{36}$$

*(i.e. $x$ at which $U_d$ is differentiable, at least one of (34) or (35) is verified with equality)*

$$U_{\tilde{d}}(x) = \left(S^{\tilde{d}}U\right)(x) \qquad \forall \tilde{d} \neq d,\ \text{if } x \in \gamma_{\tilde{d}}^{-}\ \ (d \neq 0), \tag{37}$$

$$U_d(x) = \left(S^d U\right)(x) \qquad \text{if } x \in \gamma_d^{+}\ \ d \neq 0. \tag{38}$$

The proof is immediate, taking into account (27)-(29).

## 4.4 $U$ as the maximum subsolution of H-J-B equation

We identify here the optimal cost $U$ as a maximum subsolution, as a preliminary step to define the discrete problem.

**Definition 4.2** Set of subsolutions W

$$W = \left\{w(\cdot) : D \times Q \to \Re \mid w_d(\cdot) \in W^{1,\infty}_{loc}(Q),\ (34),\ (35)\right\}. \tag{39}$$

**Theorem 4.4** *$U$ is the maximum element of the set $W$, i.e. $U \in W$ and*

$$U_d(x) \geq w_d(x) \qquad \forall x \in Q,\ \forall d \in D,\ \forall w \in W. \tag{40}$$

By virtue of this theorem conditions (36)-(38) are skipped and in its place the concept of maximum element is introduced. In this form, the computation of $U$ is transformed into the problem

$$\text{Problem P}: \textit{Find the maximum element } U \textit{ of the set } W. \tag{41}$$

# 5 The discrete problem

## 5.1 Elements of the discrete problem

To define the discrete problem, it is necessary to introduce an approximation which comprises a discretization of the space $W^{1,\infty}_{loc}(\Omega)$ and a discretization of conditions (34)-(35). We use some techniques analyzed in [3], [9].

### 5.1.1 Approximation of domain $Q$

We will approximate $Q$ with $Q_k = \bigcup_j S^k_j$, where $S^k_j$ is a finite set of quadrilateral elements and, in consequence, $Q_k$ is a polyhedron of $\Re^m$ . We define

$$k = \max_j(\mathrm{diam} S^k_j).$$

We use a special uniform mesh $B^k$ of the space $\Re^m$ . This mesh is defined in terms of an arbitrary parameter $h$, in the following way

$$B^k = \left\{ x^0 + \sum_{d=0}^{m} \varsigma_d \, e^d \ : \ \varsigma_d \text{ integer} \right\} \tag{42}$$

$$h_d = \frac{r_d}{p_d} h$$

$$h_0 = \left(1 - \sum_{d=1}^{m} \frac{r_d}{p_d}\right) h$$

$$e^0 = (-r_1, ..., -r_i, ..., -r_m)\, h_0$$

$$e^d = (-r_1, ..., -r_{d-1}, p_d - r_d, -r_{d+1}, ..., -r_m)\, h_d$$

We will say that $S^k_j$ is an elementary domain of $Q_k$ if it has the following form

$$S^k_j = x^k + \left\{ x = \sum_{d=0}^{m} \varsigma_d \, e^d \ : \ \varsigma_d \in [0,1] \right\}, \qquad x^k \in B^k, \qquad S^k_j \subset Q. \tag{43}$$

We will denote by $V^k = \{x^i \, , \, i = 1, ...N\}$ the set of nodes of $Q_k$ and we will denote the cardinal of $V^k$ by $N$. The typical shape of this mesh can be seen in Figure 1.

**Remark 5.1** If $k$ is small enough, for any two vertices of $V^k$, there always exists a path given by a natural trajectory of the system that joins the first vertex with the second one (we can see this phenomenon in Figure 2).

**Remark 5.2** From (7) and (42) it results that $B^k$ can be generated by

$$B^k = \left\{ x^0 + \sum_{d=1}^{m} \varsigma_d \, e^d \ : \ \varsigma_d \text{ is an integer} \right\}. \tag{44}$$

**Definition 5.1 Discrete controls associated to the mesh.**

We introduce a special family of controls by constraining the distance between switching times in the following way

$$A_x^{d,k} = \left\{\alpha(\cdot) \in A_x^d \ : \ \theta_{i+1} - \theta_i = \varsigma\, h_{d_i}\,, \ \ \varsigma \text{ integer}\right\}. \tag{45}$$

### An Interpretation of the Mesh

The special mesh used, originates a discrete optimal control problem. In that problem, the system has an evolution given by the differential equation (14), but controls $d_i$ are applied during intervals whose length is $\varsigma\, h_{d_i}$ and the initial state $x$ must be a node of $V^k$.
In consequence, the trajectory associated to this control reaches a node of the mesh at every switching time.
Taking into account the interpretation of the discrete equations as the optimality conditions over the Markov chain associated to this discretization, this interpretation implies that the chain is deterministic in the sense that $P_{i,j} = 0$ or $1$ (see [16]).
This property of the mesh plays a key role in relation to the precision of the method and the velocity of convergence of its computational algorithm.

### 5.1.2 Approximation of the boundary

We define, $\forall d = 1, ..., m$

$$\begin{aligned} \gamma_{k,d}^{+} &= \left\{x^i \in V^k \ : \ x^i + h_d\, g(d) \notin Q_k\right\}, \\ \gamma_{k,d}^{-} &= \left\{x^i \in V^k \ : \ x^i + h_{\hat{d}}\, g(\hat{d}) \notin Q_k, \ \forall \hat{d} \neq d\right\}. \end{aligned} \tag{46}$$

### 5.1.3 Definition of the approximation space $F^k$

We consider the set $F^k$ of functions $w : Q^k \times D \to \Re$, $w(\cdot, d) \in W^{1,\infty}(Q_k)$, such that in each quadrilateral element $Q_k$, $w(\cdot, d)$ is a polynomial that belongs to the $Q^1$ family (see [18], [8], for the corresponding definitions). It is obvious that any $w \in F^k$ is uniquely characterized by the values $w(x^i, d)$, $x^i \in V^k$, $d \in D$.

### 5.1.4 Discretization of H-J-B inequalities

We will use the following discretization of conditions (34), (35); which take forms (47), (48), respectively,

$$w(x^i, d) \leq S^d(w)(x^i) \qquad \forall x^i \in V^k, \ \forall d \in D, \tag{47}$$

$$w(x^i, d) \leq (D_d^k w)(x^i) \qquad \forall x^i \in V^k, \ \forall d \in D. \tag{48}$$

Here, $D_d^k$ is defined by:

$$\left|\begin{array}{l} (D_d^k w)(x^i) = e^{-\lambda h_d}\, w(x^i + h_d\, g(d), d) + \int\limits_0^{h_d} f(x^i + s\, g(d), d)\, e^{-\lambda s}\, ds \\ \qquad\qquad\qquad\qquad \forall x^i \in \left(V^k \bigcap C\gamma_{k,d}^+ \bigcap C\left(\bigcup_{r \neq d} \gamma_{k,d}^-\right)\right), \\ (D_d^k w)(x^i) = +\infty \qquad\qquad \forall x^i \in \left(\gamma_{k,d}^+ \bigcup \left(\bigcup_{r \neq d} \gamma_{k,d}^-\right)\right). \end{array}\right. \tag{49}$$

**Remark 5.3** We can see that the definition of $D_d^k$ is consistent with (35) and that it also takes into account the boundary constraints (37)-(38).

**Remark 5.4** We can use (50) instead of (49):

$$\begin{gathered} (D_d^k w)(x^i) = e^{-\lambda h_d}\, w(x^i + h_d g(d), d) + h_d\, f(x^i, d)\;, \\ \forall x^i \in \left(V^k \bigcap C\gamma_{k,d}^+ \bigcap C\left(\bigcup_{r \neq d} \gamma_{k,d}^-\right)\right). \end{gathered} \tag{50}$$

It can be easily proved that the corresponding associated discrete problem has the same convergence properties as the discrete problem $\mathrm{P}_k$ defined below.

### 5.1.5 Definition of operator $P_k$

We define the operator $P_k : F_k \to F_k$ as

$$(P_k\, w)(x^i, d) = \min\left((D_d^k w)(x^i),\ (S_d(w))(x^i)\right), \qquad \forall x^i \in V^k,\ \forall d \in D. \tag{51}$$

### 5.1.6 Definition of discrete subsolutions and supersolutions

The set $\mathrm{W}_k$ of discrete subsolutions is defined by

$$W_k = \left\{w(\cdot,\cdot) \in F_k \ :\ w(x^i, d) \le (P_k\, w)(x^i, d),\ \forall x^i \in V^k,\ \forall d \in D\right\}. \tag{52}$$

In a similar way we define the set $S_k$ of discrete supersolutions

$$S_k = \left\{s(\cdot,\cdot) \in F_k \ :\ s(x^i, d) \ge (P_k\, s)(x^i, d),\ \forall x^i \in V^k,\ \forall d \in D\right\}. \tag{53}$$

## 5.2 Three equivalent discrete problems

In relation to the original problem P we introduce the discrete problems:

$$\text{Problem } \mathrm{P}_k^1 : \textit{Find the maximum element of } W_k \tag{54}$$

$$\text{Problem } \mathrm{P}_k^2 : \textit{Find the fixed point of operator } P_k \tag{55}$$

$$\text{Problem } \mathrm{P}_k^3 : \textit{Find the minimum element of } S_k \tag{56}$$

By using basically the techniques introduced in [15], we can prove the following result:

**Problems $\mathbf{P}_k^1$, $\mathbf{P}_k^2$ and $\mathbf{P}_k^3$ are equivalent**

**in the sense that they have the same unique solution $U^k$.**

The equivalence of problems $\mathbf{P}_k^1$, $\mathbf{P}_k^2$ and $\mathbf{P}_k^3$ and a wide characterization of the unique solution $U^k$ are given by the following theorem.

**Theorem 5.1**

- $\exists$ *unique maximum element (the maximum subsolution) of $W^k$ defined by*

$$\overline{U}^k(x^i, d) = \sup_{w_d \in W_k} \Big( w(x^i, d) \Big) \tag{57}$$

- $\overline{U}^k$ *is a fixed point of $P_k$ i.e.*

$$\overline{U}^k = P_k \overline{U}^k \tag{58}$$

- $\exists$ *unique minimum element (the minimum supersolution) of $S_k$ defined by*

$$\underline{U}^k(x^i, d) = \inf_{s_d \in S_k} \Big( s(x^i, d) \Big) \tag{59}$$

- $\underline{U}^k$ *is a fixed point of $P_k$, i.e.*

$$\underline{U}^k = P_k \underline{U}^k \tag{60}$$

- $\exists$ *unique fixed point of $P_k$, i.e.*

$$U^k = P_k U^k \tag{61}$$

- *$U^k$ is the minimum supersolution and the maximum subsolution, i.e.*

$$\{U^k\} = S_k \bigcap W_k$$

*and*

$$\underline{U}^k = \overline{U}^k = U^k = P_k U^k.$$

- *$U^k$ can be computed with the following iterative algorithm which converges from any starting point $U^0$.*

**Algorithm $\mathbf{A}_0$***:*

*Step 0: Set $\nu = 0,\ U^0 \in F_k$*

*Step 1: Set $U^{\nu+1} = P_k U^\nu$*

*Step 2: If $U^{\nu+1} = U^\nu$, stop; else $\nu = \nu + 1$, and go to step 1.*

- *$\forall w \in W_k$, $w^\nu$ recursively defined by $w^{\nu+1} = P_k w^\nu$, $w^0 = w$, verifies*

$$\begin{aligned} &w^{\nu+1} \geq w^\nu, \\ &w^\nu \to U^k. \end{aligned} \tag{62}$$

- *$\forall s \in S_k$, $s^\nu$ recursively defined by $s^{\nu+1} = P_k s^\nu$, $s^0 = s$, verifies*

$$\begin{aligned} &s^{\nu+1} \leq s^\nu, \\ &s^\nu \to U^k. \end{aligned} \tag{63}$$

- *the following rate of convergence holds*

$$\left\| P_k w - U^k \right\| \leq K(\|w\|)\ (1 - \mu(\|w\|))^\nu, \tag{64}$$

*where $0 < \mu(\|w\|) \leq 1$ and $K(\|w\|) > 0$.*

**Remark 5.5** The proof of Theorem 5.1 follows the arguments introduced in [15] to prove the convergence of Bensoussan-Lions Algorithm for QVI.

**Remark 5.6** The real computation of $U^k$ is done in practice with more efficient algorithms, like those presented in section 6.

### 5.3 Convergence of the solutions

Below, we obtain an estimation of the difference between the continuous function $U$ - the minimum that can be reached with the original policies of $A^d_x$ - and the discrete solution $U^k$ - i.e. the minimum that can be reached with the discrete policies $A^{d,k}_x$, defined in (45).

From this estimation, it follows that the convergence of $U^k$ to $U$ is (locally) of order $k$. The convergence is uniform in closed subsets of $Q$ not intersecting $\partial Q^+$.

To get this estimation, we will prove here that any optimal policy can be approximated, with an error of $k$-order, using an element of $A^{d,k}_x$.

**Notation:**

Here, $[r]$ denotes the integer part of an arbitrary real number $r$. We define:

$$\eta = \max_{i,d} \left| \frac{1}{g_i(d)} \right|, \qquad \hat{\eta} = \eta \, M_g \,, \tag{65}$$

$$\hat{\nu}_x = \left[ \overline{C} \left( 1 + \left( \log(d(x, \partial Q^+)) \right)^- \right) \right] + 1 \,, \tag{66}$$

$$\hat{c} = \max\{ h_d / k \ : \ d \in D \} \,, \tag{67}$$

$$M = \frac{1 + \hat{\eta}}{\hat{\eta}} (2 \hat{\eta} + M_g \, \hat{c}) \,, \tag{68}$$

$$C(\nu) = M \, \frac{(1 + \hat{\eta})^{\nu+1}}{\hat{\eta}} + 2 \eta \, \nu + \nu \, \hat{c} \,, \tag{69}$$

$$\tilde{C}(d(x^k, \partial Q^+)) = \sum_{\nu=1}^{\nu_x} C(\nu) \,. \tag{70}$$

**Proposition 5.1** *Let $x^k \in V^k$, $d \in D$ and $\bar{\alpha}(\cdot)$ an optimal policy for the initial conditions $(x^k, d)$, with $y(\cdot)$ its corresponding trajectory. Then there exists a discrete control $\alpha^k \in A_x^{d,k}$ such that, if we denote with $y^k$ its associated trajectory, it is verified:*

$$\left\|y^k(s) - y(s)\right\| \leq M\,(1+\hat{\eta})^{\hat{\nu}_x+1}\ k, \qquad \forall s \in [0,T], \tag{71}$$

$$m\Big\{s \in [0,T]\ :\ \alpha(s) \neq \alpha^k(s)\Big\} \leq \tilde{C}(d(x^k, \partial Q^+))\,k\,, \tag{72}$$

*and if $\theta_\nu^k$ are the switching times of $\alpha^k$, then*

$$\begin{cases} \theta_\nu^k \leq \theta_\nu\,, \\ \theta_\nu - \theta_\nu^k \leq C(\nu)\,k\,, \end{cases} \tag{73}$$

**Proof.** For the sake of simplicity, the proof will be restricted to the case $m = 2$.

Let $T$ be given by (19). Theorem 3.1 implies that $\bar{\alpha}$ has at most $\nu_x$ switching times $\theta_1, ..., \theta_{\nu_x}$ in $[0, T]$, $\nu_x \leq \hat{\nu}_x$, where $\hat{\nu}_x$ is given by (66). This estimation is independent on $\bar{\alpha}$, (it only depends on $\|x\|$).

Following the method used in [10], we define a discrete control policy $\alpha^k(\cdot)$, selecting the switching times $\theta_1^k, ..., \theta_{\nu_x}^k$ in such a form that the discrete trajectory obtained is admissible.

We define recursively, the switching points in the following way

$$\theta_0^k = \theta_0\,, \tag{74}$$

$$\theta_{\nu+1}^k = \theta_\nu^k + \left[\frac{((\theta_{\nu+1} - \theta_\nu) - \Delta_\nu)^+}{h_{d_\nu}}\right] h_{d_\nu}\,. \tag{75}$$

$$y^k(\theta_{\nu+1}^k) = y^k(\theta_\nu^k) + g(d_\nu)\,(\theta_{\nu+1}^k - \theta_\nu^k)\,. \tag{76}$$

$$\delta_\nu = \max_i \left|y_i(\theta_\nu) - y_i^k(\theta_\nu^k)\right|\,, \tag{77}$$

$$\Delta_\nu = \eta\,(\delta_\nu + 2\,k)\,, \tag{78}$$

$$\mu_\nu = \theta_\nu - \theta_\nu^k\,. \tag{79}$$

We must see that $\alpha^k$ is admissible, i.e. $y^k(\theta_{\nu+1}^k) \in V^k$, $\forall\,\nu = 0, 1, ....$

For $\nu = 0$, clearly $y^k(\theta_0^k) = y(\theta_0) \in V^k$. For $\nu = 1, \ldots$, we proceed by induction. Then, let us suppose $y^k(\theta_\nu^k) \in V^k$. When

$$\left[\frac{((\theta_{\nu+1} - \theta_\nu) - \Delta_\nu)^+}{h_{d_\nu}}\right] = 0,$$

it is $\theta_{\nu+1}^k = \theta_\nu^k$ and in consequence, from (76), $y^k(\theta_{\nu+1}^k) \in V^k$.

Now we consider

$$\left[\frac{((\theta_{\nu+1} - \theta_\nu) - \Delta_\nu)^+}{h_{d_\nu}}\right] \geq 1. \tag{80}$$

We will see that it is verified:

$$\text{if} \quad y_i^k(\theta_{\nu+1}^k) > y_i^k(\theta_\nu^k) \quad \text{then} \quad 0 \leq y_i^k(\theta_{\nu+1}^k) \leq M_i - 2\,k\,, \tag{81}$$

$$\text{if} \quad y_i^k(\theta_{\nu+1}^k) < y_i^k(\theta_\nu^k) \quad \text{then} \quad 2\,k \leq y_i^k(\theta_{\nu+1}^k) \leq M_i\,. \tag{82}$$

It is obvious that (81)-(82) imply $y^k(\theta_{\nu+1}^k) \in V^k$.

For (81)-(82), we analyze the different possible cases. There exist two principal cases

- $y_i(\theta_\nu) \geq y_i^k(\theta_\nu^k)$ and $g_i(d_\nu) > 0$

  In consequence

$$\begin{aligned} 0 \leq y_i^k(\theta_{\nu+1}^k) &= y_i^k(\theta_\nu^k) + g_i(d_\nu)\,(\theta_{\nu+1}^k - \theta_\nu^k) \\ &\leq y_i(\theta_\nu) + g_i(d_\nu)\,(\theta_{\nu+1} - \theta_\nu) - 2\,k = y_i(\theta_{\nu+1}) - 2\,k \leq M_i - 2\,k\,, \end{aligned}$$

  then, (81) is verified.

- $y_i(\theta_\nu) \geq y_i^k(\theta_\nu^k)$ and $g_i(d_\nu) < 0$

  By induction, obviously it holds

$$y_i^k(\theta_{\nu+1}^k) < y_i^k(\theta_\nu^k) \leq M_i$$

  and by virtue of (80)

$$(\theta_{\nu+1} - \theta_\nu) > \Delta_\nu\,.$$

Then

$$
\begin{aligned}
y_i^k(\theta_{\nu+1}^k) &= y_i^k(\theta_\nu^k) + g_i(d_\nu)\left[\tfrac{((\theta_{\nu+1}-\theta_\nu)-\Delta_\nu)^+}{h_{d_\nu}}\right] h_{d_\nu} \\
&> y_i^k(\theta_\nu^k) + g_i(d_\nu)(\theta_{\nu+1}-\theta_\nu) - g_i(d_\nu)\,\eta\,(\delta_\nu + 2\,k) \\
&\geq y_i(\theta_\nu) - \delta_\nu + g_i(d_\nu)(\theta_{\nu+1}-\theta_\nu) - g_i(d_\nu)\,\eta\,(\delta_\nu + 2\,k) \\
&\geq y_i(\theta_\nu) + g_i(d_\nu)(\theta_{\nu+1}-\theta_\nu) + 2\,\eta\,k \\
&= y_{i+1}(\theta_\nu) + 2\,\eta\,k \geq 2\,\eta\,k\,.
\end{aligned}
$$

The remaining cases are proved in similar form, which allows us to obtain an admissible discrete policy $\alpha^k(\cdot)$.

Let us calculate an estimation of $(\delta_\nu,\, \mu_\nu)$. The initial values are

$$\delta_0 = 0, \qquad \mu_0 = 0.$$

From (75) it results

$$\theta_{\nu+1}^k \leq \theta_{\nu+1}\,, \tag{83}$$

$$\left|(\theta_{\nu+1}^k - \theta_\nu^k) - (\theta_{\nu+1} - \theta_\nu)\right| \leq \eta\,\delta_\nu + 2\,\eta\,k + h_{d_\nu}\,, \tag{84}$$

$$\theta_{\nu+1} - \theta_{\nu+1}^k = (\theta_{\nu+1} - \theta_\nu) + (\theta_\nu^k - \theta_{\nu+1}^k) + (\theta_\nu - \theta_\nu^k) \leq \eta\,(\delta_\nu + 2\,k) + h_{d_\nu} + \mu_\nu \tag{85}$$

and in consequence

$$\mu_{\nu+1} \leq \mu_\nu + \eta\,\delta_\nu + 2\,\eta\,k + h_{d_\nu}\,. \tag{86}$$

For a generic component $i$, we have

$$
\begin{aligned}
y_i(\theta_{\nu+1}) &= y_i(\theta_\nu) + g_i(d_\nu)\,(\theta_{\nu+1} - \theta_\nu)\,, \\
y_i^k(\theta_{\nu+1}^k) &= y_i^k(\theta_\nu^k) + g_i(d_\nu)\,(\theta_{\nu+1}^k - \theta_\nu^k)\,,
\end{aligned}
$$

then

$$\left|y_i(\theta_{\nu+1}) - y_i^k(\theta_{\nu+1}^k)\right| \leq \left|y_i(\theta_{\nu 1}) - y_i^k(\theta_\nu^k)\right| + M_g\,\left|(\theta_{\nu+1}^k - \theta_\nu^k) - (\theta_{\nu+1} - \theta_\nu)\right|. \tag{87}$$

From (84) and (87), we have

$$\delta_{\nu+1} \leq \delta_\nu\,(1+\hat{\eta}) + 2\,\hat{\eta}\,k + M_g\,h_{d_\nu}$$

and in consequence (using the definition (68)) we obtain:

$$\delta_\nu \leq M\,(1+\hat{\eta})^\nu k\,. \tag{88}$$

Also, taking into account (67), (86), (88), we get

$$\mu_{\nu+1} \leq \frac{M}{\hat{\eta}}\,\eta\,(1+\eta)^{\nu+1} + 2\,\eta\,\nu\,k + \nu\,\hat{c}\,k\,,$$

i.e., using (69),

$$\mu_{\nu+1} \leq C(\nu)\,k\,.$$

Finally

$$\left\{s \in [0,T] \;:\; \alpha(s) \neq \alpha^k(s)\right\} \subseteq \bigcup_{\nu=1}^{\nu_x} [\theta_\nu^k,\, \theta_\nu]\,,$$

then

$$\mathrm{m}\left\{s \in [0,T] \;:\; \alpha(s) \neq \alpha^k(s)\right\} \leq \sum_{\nu=1}^{\nu_x} \mu_\nu \leq \tilde{C}(\|x\|)\,k\,.$$

□

With this result we can obtain the following estimations.

**Theorem 5.2** *Let $\epsilon > 0$, and $K_\epsilon = \{x \in Q \;:\; \|x\| \geq \epsilon\}$, then there exists $C_2(\epsilon) \in \Re^+$ such that*

$$U_d^k(x^i) - U_d(x^i) \leq \frac{C_2(\epsilon)}{\lambda}\,k, \qquad \forall x^i \in V^k \bigcap K_\epsilon\,,\ \forall d \in D.$$

**Proof.** Let $x_0^k \in V^k$, $d_0 \in D$ such that

$$U_{d_0}^k(x_0^k) - U_{d_0}(x_0^k) = \max\left\{U_d^k(x^k) - U_d(x^k) \;:\; x^k \in Q^k \bigcap \mathrm{K}_\epsilon\,,\ d \in D\right\}$$

and $\bar{\alpha}(\cdot)$ an optimal policy corresponding to the initial condition $(x_0^k, d_0)$.

By virtue of Theorems 3.1 and 3.5, there exists $c_m > 0$, $\theta_{\bar{\nu}} \in [0,T]$ such that

$$\alpha(\theta_{\bar{\nu}}) = 0,$$

$$\theta_{\bar{\nu}+1} - \theta_{\bar{\nu}} \geq \frac{2\,c_m}{1 + (\log\epsilon)^-}\,.$$

This property also implies, for $\epsilon$ small enough, that $y(\theta^+_{\bar\nu}) \in K_\epsilon$, where

$$\theta^+_{\bar\nu} = \frac{\theta_{\bar\nu+1} + \theta_{\bar\nu}}{2},$$

$$\theta^+_{\bar\nu} \geq \frac{c_m}{1 + (\log \epsilon)^-}.$$

Let us construct $\alpha^k$ taking into account Proposition 5.1. We define

$$\theta^{+,k}_{\bar\nu+1} = \theta^k_{\bar\nu} + \left[\frac{\theta^k_{\bar\nu+1} - \theta^k_{\bar\nu}}{2\, h_0}\right] h_0. \tag{89}$$

So, by construction it results $y^k(\theta^{+,k}_{\bar\nu+1}) \in Q^k$ and

$$\theta^{+,k}_{\bar\nu} \geq \frac{c_m}{1 + (\log \epsilon)^-}. \tag{90}$$

By virtue of the dynamic programming principle for $U^k$ and $U$, we have

$$U^k_{d_0}(x^k_0) \leq \int_0^{\theta^{+,k}_{\bar\nu}} f(y^k(t), \alpha^k(t))\, e^{-\lambda t}\, dt + \sum_{i=0}^{\bar\nu} q(d_i, d_{i+1})\, e^{-\lambda\, \theta^k_i} + U^k_0(y^k(\theta^{+,k}_{\bar\nu}))\, e^{-\lambda\, \theta^{+,k}_{\bar\nu}}$$

and

$$U_{d_0}(x^k_0) = \int_0^{\theta^+_{\bar\nu}} f(y(t), \alpha(t))\, e^{-\lambda t}\, dt + \sum_{i=0}^{\bar\nu} q(d_i, d_{i+1})\, e^{-\lambda\, \theta_i} + U_0(y(\theta^{+,k}_{\bar\nu}))\, e^{-\lambda\, \theta^+_{\bar\nu}}.$$

Let us denote

$$I_c = \int_0^{\theta^{+,k}_{\bar\nu}} f(y^k(t), \alpha^k(t))\, e^{-\lambda t}\, dt - \int_0^{\theta^+_{\bar\nu}} f(y(t), \alpha(t))\, e^{-\lambda t}\, dt,$$

$$I_q = \sum_{i=0}^{\bar\nu} q(d_i, d_{i+1})\, e^{-\lambda\, \theta^k_i} - \sum_{i=0}^{\bar\nu} q(d_i, d_{i+1})\, e^{-\lambda\, \theta_i}.$$

By Proposition 5.1 we have the following estimations

$$0 \leq \theta_\nu - \theta^k_\nu \leq C(\nu)\, k \qquad \forall \nu \leq \bar\nu, \tag{91}$$

$$\theta^+_{\bar\nu} - \theta^{+,k}_{\bar\nu} \leq C(\bar\nu)\, k + c\, k, \tag{92}$$

$$\mathrm{m}\Big\{s \in [0,T] \;:\; \alpha(s) \neq \alpha^k(s)\Big\} \leq \widetilde{C}(\|x\|)\, k. \tag{93}$$

Then, from the hypotheses on $f$, using Proposition 5.1 and (92)-(93) it results

$$I_c \leq \widetilde{M}(\epsilon)\, k\,, \qquad I_q \leq \widetilde{M}(\epsilon)\, k\,.$$

In consequence, from Proposition 5.1 we obtain:

$$\begin{aligned}
U^k_{d_0}(x^k_0) - U_{d_0}(x^k_0) &\leq U^k_0(y^k(\theta^{+,k}_{\bar\nu}))\, e^{-\lambda\,\theta^{+,k}_{\bar\nu}} - U_0(y(\theta^+_{\bar\nu}))\, e^{-\lambda\,\theta^+_{\bar\nu}} + 2\,\widetilde{M}(\epsilon)\, k \\
&\leq e^{-\lambda\,\theta^{+,k}_{\bar\nu}} \left(U^k_0(y^k(\theta^{+,k}_{\bar\nu})) - U_0(y^k(\theta^{+,k}_{\bar\nu}))\right) \\
&\quad + \left(e^{-\lambda\,\theta^{+,k}_{\bar\nu}} - e^{-\lambda\,\theta^+_{\bar\nu}}\right) U_0(y^k(\theta^{+,k}_{\bar\nu})) \\
&\quad + e^{-\lambda\,\theta^+_{\bar\nu}} \left(U_0(y^k(\theta^{+,k}_{\bar\nu})) - U_0(y(\theta^{+,k}_{\bar\nu}))\right) \\
&\quad + e^{-\lambda\,\theta^+_{\bar\nu}} \left(U_0(y(\theta^{+,k}_{\bar\nu})) - U_0(y(\theta^+_{\bar\nu}))\right) + 2\,\widetilde{M}(\epsilon)\, k \\
&\leq e^{-\lambda\,\theta^+_{\bar\nu}} \left(U^k_0(x^k_0) - U_0(x^k_0)\right) + C_U\,(1 + \lambda + (\log(\epsilon))^-)\, C(\hat\nu_x)\, k \\
&\quad + L(\epsilon)\, M\,(1+\hat\eta)^{\hat\nu_x+1}\, k + L(\epsilon)\, M_g\, C(\hat\nu_x)\, k + 2\,\widetilde{M}(\epsilon)\, k\,.
\end{aligned}$$

Then

$$\left(U^k_{d_0}(x^k_0) - U_{d_0}(x^k_0)\right)\left(1 - e^{-\lambda\,\theta^{+,k}_{\bar\nu}}\right) \leq C_1(\epsilon)\, k$$

and in consequence

$$U^k_{d_0}(x^k_0) - U_{d_0}(x^k_0) \leq \frac{C_1(\epsilon)\, k}{1 - e^{-\lambda\,\theta^{+,k}_{\bar\nu}}} \leq \frac{C_2(\epsilon)}{\lambda}\, k\,,$$

because $\theta^{+,k}_{\bar\nu}$ verifies (90).

□

**Corollary 5.1** *The discretization error can be estimated in the following form*

$$\left\| U - U^k \right\| \leq \frac{C_2(\epsilon)}{\lambda}\, k\,. \tag{94}$$

**Proof.** As $A^{d,k}_x \subset A^d_x$, we have

$$U \leq U^k\,;$$

this, together with Theorem 5.2 implies that (94) holds.

□

# 6 Numerical algorithm

We define here a fast algorithm which is a combination of value iteration algorithm and policy iteration algorithm.

## 6.1 Preliminary definitions

- Set of multi-valued or generalized discrete policies

$$\Lambda = \left\{ A : V^k \times D \to 2^D \right\}$$

- Set of mono-valued discrete policies

$$\Theta = \left\{ A \in \Lambda \; : \; \forall (x^i, d) \in V^k \times D, \; \mathrm{card}(A(x^i, d)) = 1 \right\}$$

- Linear system associated to a feedback discrete policy $\tilde{A} \in \Theta$.

  We consider a linear system that we denote in a brief way

$$Lu = b. \tag{95}$$

  For this system, the relation that defines a generic $i$-th equation is

$$\left| \begin{array}{ll} u(x^i, d) = (D_d^k w)(x^i) & \text{if} \quad \tilde{A}(u)(x^i, d) = d\,, \\ & \\ u(x^i, d) = q(d, \tilde{d}) + u(x^i, \tilde{d}) & \text{if} \quad \tilde{A}(u)(x^i, d) = \tilde{d}\,. \end{array} \right.$$

- $\epsilon-$suboptimal multi-valued discrete controls associated to $w$ : $A_\epsilon w$

$$\left| \begin{array}{l} A_\epsilon \in \Lambda\,, \\ \\ (A_\epsilon w)(x^i, d) = (B_\epsilon w)(x^i, d) \bigcap (C_\epsilon w)(x^i, d)\,, \end{array} \right. \tag{96}$$

  where

$$(B_\epsilon w)(x^i, d) = \left\{ \tilde{d} \in D \; : \; (P_k w)(x^i, d) + \epsilon \geq q(d, \tilde{d}) + w(x^i, \tilde{d}) \right\},$$

$$(C_\epsilon w)(x^i, d) = \left| \begin{array}{lll} \{d\} & \text{if} & (P_k w)(x^i, d) \geq (D_d^k w)(x^i) - \epsilon\,, \\ & & \\ \emptyset & \text{if} & (P_k w)(x^i, d) < (D_d^k w)(x^i) - \epsilon\,. \end{array} \right.$$

## 6.2 The accelerated algorithm

The algorithm uses a combination of value iteration algorithm and policies iteration algorithm. It is a modification of the methodology presented in [11], [14].

Step 1: Set $w^{0,0} \in F_k, \; \bar{r} > 0.$

Step 2: $\nu = 0, \; \eta = 0, \; A^{\nu} = \{\emptyset\}^{(m+1)\times N}.$

Step 3: $\eta = \eta + 1$, compute $w^{\nu,\eta} = P_k(w^{\nu,\eta-1}), \; A^{\nu,\eta} = A_\epsilon(w^{\nu,\eta}).$

Step 4: If $A^{\nu,\eta} = A^{\nu}, \; r = r + 1;$

else, $r = 0, \; A^{\nu} = A^{\nu,\eta}$, and go to Step 3.

Step 5: If $r \geq \bar{r}$, choose any $A \in \Theta$ such that $\forall (x^i, d) \; A(x^i, d) \subset A^{\nu,\eta}(x^i, d)$ and construct the system $L^{\nu,\eta}u = b^{\nu,\eta}$;

else, go to Step 3.

Step 6: If $\det(L^{\nu,\eta}) \neq 0$, compute the solution $u$ of the system $L^{\nu,\eta}u = b^{\nu,\eta}$;

else, $r = 0, \; A^{\nu} = \{\emptyset\}^{(m+1)\times N}$ and go to Step 3.

Step 7: If $u = P_k u$, set $U^k = u$, end;

else, go to Step 8.

Step 8: If $(\; \nu = 0$ or $u \leq w^{\nu,\eta} \;)$, $w^{\nu+1,0} = u, \; \nu = \nu + 1, \eta = 0, \; r = 0,$ $A^{\nu} = \{\emptyset\}^{(m+1)\times N}$, and go to Step 3;

else, $\epsilon = \gamma\epsilon$, and go to Step 3.

## 6.3 Convergence of the algorithm

The algorithm has a fast convergence; in fact, as the set of discrete policies is finite, the algorithm has the following property (the proof can be seen in [10]).

**Theorem 6.1** *The algorithm converges in a finite number of steps.*

## 6.4 Practical resolution of the linear systems associated to the algorithm

The linear systems $L^{\nu,\eta}u = b^{\nu,\eta}$ at Step 6 can be easily solved by virtue of the simple structure of the $L^{\nu,\eta}$ matrix.

In fact, using a suitable rearrangement of the indices, we can write this matrix in a lower block triangular form, specifically having the following shape

$$\begin{bmatrix} D_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & D_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \cdots & \cdots & \ddots & & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & D_r & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ C_{1,1} & C_{1,2} & \cdots & C_{1,r_1} & T_1 & \cdots & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ C_{s,1} & C_{s,2} & \cdots & C_{s,r_1} & C_{s,r_1+1} & \cdots & C_{s,r_s} & T_s \end{bmatrix},$$

where, for each $\delta = 1, ..., r$ the block $D_\delta$ has the form

$$\begin{bmatrix} 1 & l^{\nu,\eta}_{\delta,1,2} & 0 & \cdots & 0 \\ 0 & 1 & l^{\nu,\eta}_{\delta,2,3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & l^{\nu,\eta}_{\delta,p_\delta-1,p_\delta} \\ l^{\nu,\eta}_{\delta,1,p_\delta} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and, for each $t = 1, ..., s$, the block $T_t$ has the form

$$\begin{bmatrix} 1 & l^{\nu,\eta}_{t,1,2} & 0 & \cdots & 0 \\ 0 & 1 & l^{\nu,\eta}_{t,2,3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & l^{\nu,\eta}_{t,p_t-1,p_t} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The matrix $[C_{i,1}, C_{i,2}, ..., C_{i,i-1}]$ has all its elements equal to zero except an unique non zero element in the last row, that we denote $c_{i,j}$ from now on, being $j$ the column with the non zero element.

The linear equation restricted to the $u^\delta$ elements corresponding to the $D_\delta$ block has the form

$$u^\delta_\zeta = l^{\nu,\eta}_{\delta,\zeta,\zeta+1}\, u^\delta_{\zeta+1} + b^{\nu,\eta}_{\tilde{\zeta}},$$

being $\tilde{\varsigma} = \sum\limits_{j=1}^{s_\delta - 1} s_j + \varsigma$.
From this equation, we can deduce that the first component of the solution is

$$u_1^\delta = \frac{\sum\limits_{i=1}^{p} \left( \prod\limits_{j=2}^{i} l_{\delta,j-1,j}^{\nu,\eta} \right) b_{\tilde{i}}^{\nu,\eta}}{1 - \left( \prod\limits_{i=2}^{p} l_{\delta,i-1,i}^{\nu,\eta} \right) l_{\delta,p,1}^{\nu,\eta}}, \tag{97}$$

being $\tilde{i} = \sum\limits_{j=1}^{s_\delta - 1} s_j + i$.

The right hand side of (97) can be computed without problems because for the blocks $D_\delta$, the condition $\det(L^{\nu,\eta}) \neq 0$ implies

$$\left( \prod_{i=2}^{p} l_{\delta,i-1,i}^{\nu,\eta} \right) l_{\delta,p_\delta,1}^{\nu,\eta} < 1.$$

For the last component, we have

$$u_p^\delta = l_{\delta,p_\delta,1}^{\nu,\eta}\, u_1^\delta + b_{\tilde{p}_\delta}^{\nu,\eta}$$

and recursively

$$u_\zeta^\delta = l_{\zeta,\zeta+1}^{\nu,\eta}\, u_{\zeta+1}^\delta + b_{\tilde{\zeta}}^{\nu,\eta} \qquad \text{for } \zeta = p_\delta - 1, \ldots 2.$$

For the elements corresponding to the $T_t$ blocks, the solution is obtained in an explicit form, in terms of the solution associated to the $D_\delta$.

Obviously, for the last $u_\delta$ element appearing in a $T_t$ block, we have

$$u_p^t = c_{p_t,j}\, u_j^t + b_{\tilde{p}_t}^{\nu,\eta}.$$

For the remaining elements, recursively we have

$$u_\zeta^i = l_{\zeta,\zeta+1}^{\nu,\eta}\, u_{\zeta+1}^i + b_{\tilde{\zeta}}^{\nu,\eta}.$$

So, we see that the solution of the linear system is obtained in terms of elementary operations. Then, it is clear that the procedure can be easily programed.

# 7 Applications

We have applied the above presented numerical procedure to an example with $m = 2$ items and $\lambda = 0.0198$, $h_1 = 0.017$, $r_1 = r_2 = 1$.
The instantaneous cost function is linear in both variables and it does not depend on the parameter d, i.e. $f(x_1, x_2) = F_1 x_1 + F_2 x_2$

| | Example 1 | Example 2 | Example 3 |
|---|---|---|---|
| $M_1$ | 0.833 | 2.166 | 0.833 |
| $M_2$ | 0.833 | 2.166 | 0.833 |
| $p_1$ | 1.50 | 2.50 | 1.50 |
| $p_2$ | 6 | 6 | 6 |
| $q_{0,1}$ | 6 | 3 | 0.60 |
| $q_{0,2}$ | 15 | 15 | 15 |
| $q_{1,0}$ | 0 | 0 | 0 |
| $q_{1,2}$ | 15 | 15 | 15 |
| $q_{2,0}$ | 0 | 0 | 0 |
| $q_{2,1}$ | 6 | 3 | 0.60 |
| $F_1$ | 80 | 180 | 180 |
| $F_2$ | 10 | 80 | 10 |
| Mesh | 50× 50 | 50× 50 | 50× 50 |
| computational time | 5.76 sec. | 11.42 sec. | 3.89 sec. |

Table 1:

The results for the optimal trajectories obtained are shown in Figures 3, 4 and 5. The computational times correspond to a PC 486 computer. It can be seen the high performance of the algorithm presented in this work since the same problem (with 30×30 points) needs a computational time of 10.8 sec. using the algorithm denoted with $A_1$ in [10] and an usual triangulation in a supercomputer NEC SX-1E.

# 8 Conclusions

The methodology analyzed in this work to solve numerically the problem of optimal schedule of multi-item single machines gives an efficient procedure to calculate its solution in a short time and with a $k$-order precision.

The high velocity of the method is due to two reasons.

- One of them is the convergence of the iterative algorithm in a finite number of steps.
- The other one is that we obtain the solution of the linear systems in a explicit form and using elemental operations, instead of inverting a matrix at each step, as it is usual in policy iteration procedures.

Taking into account the precision, the method is better than those ones that use finite elements, finite differences or Markov's chains (see [3], [14], [16], [17], [20]), where precision is $\sqrt{k}$ -order.

The low precision of those methods stems from the fact that, the original process – given by a deterministic trajectory – is replaced in the approximation with Markov's chains by an stochastic process that presents, after a time T, a dispersion of $\sqrt{k}$-order around the original trajectory.
We must remark that in the approximations with finite differences or finite elements, if a *Maximum Discrete Principle* (see [5]) is satisfied, then there also exists an interpretation of the associated equations as the optimality conditions of an associated Markov's chain. In the method that we have presented, the Markov's chain is actually deterministic (i.e. the transition probabilities are $P_{i,j} = 0$ ó 1) and the approximating procedure does not present the dispersion phenomenon.

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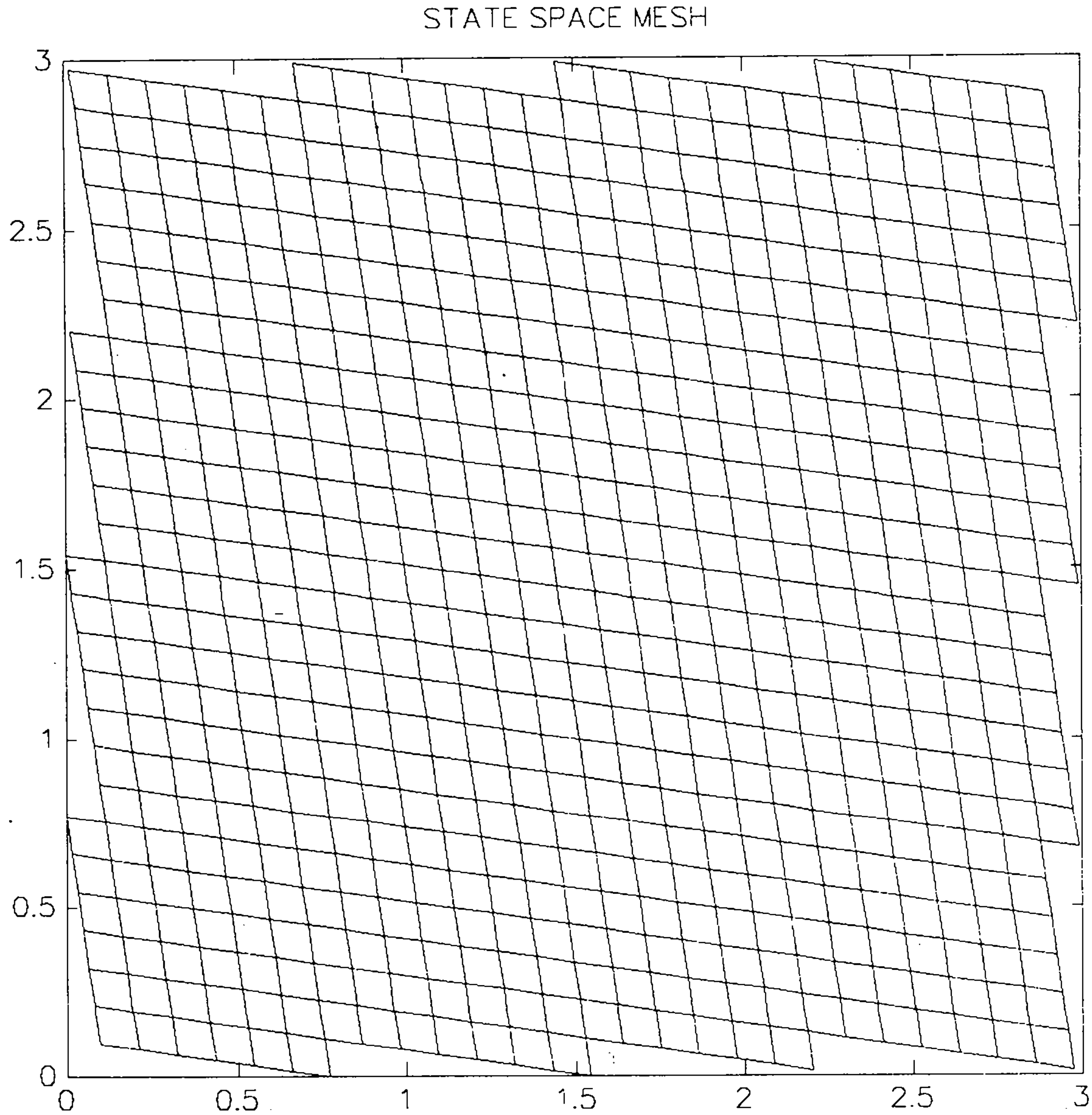
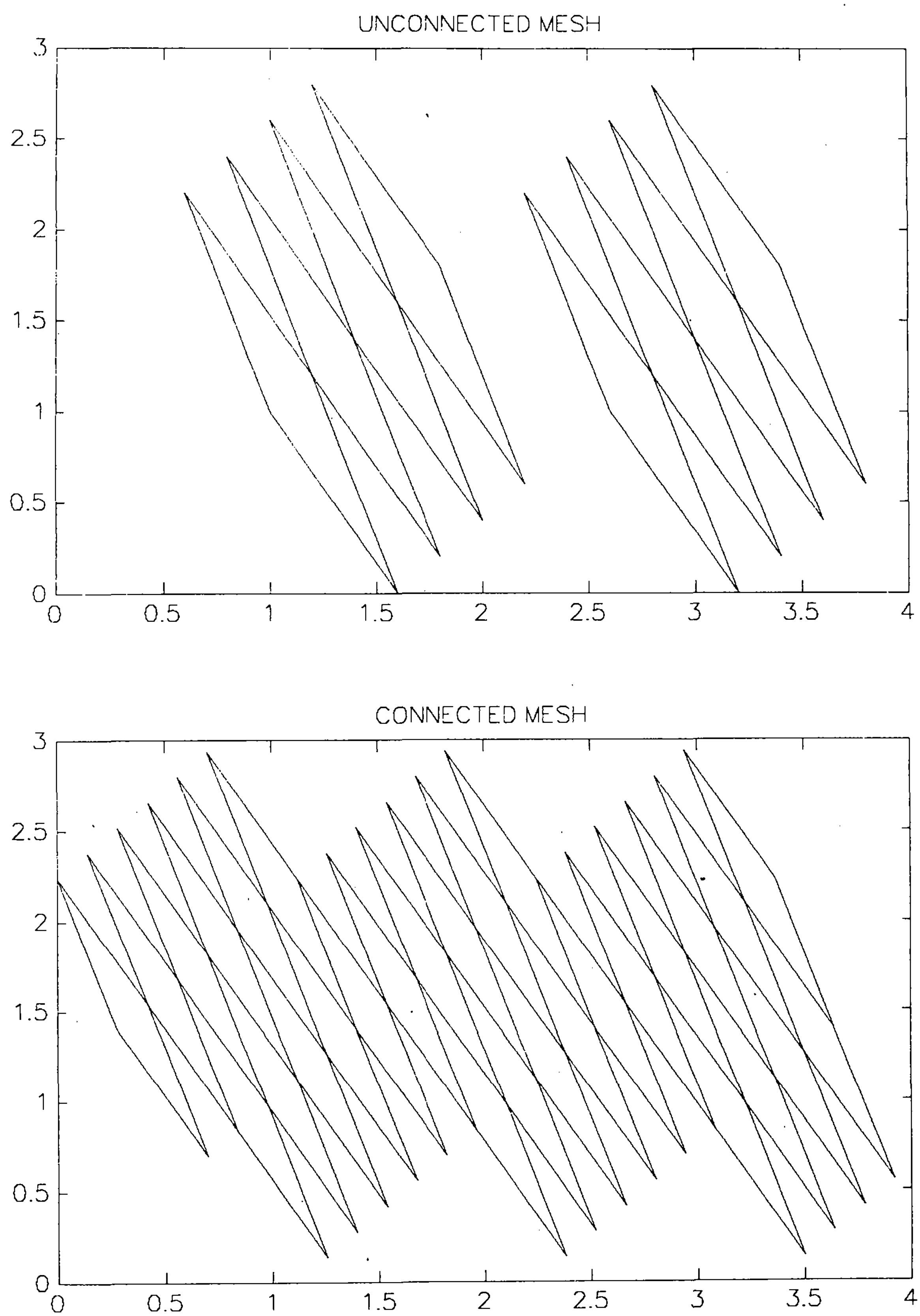



FIGURE 1



UNCONNECTED MESH
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
CONNECTED MESH
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4


FIGURE 2

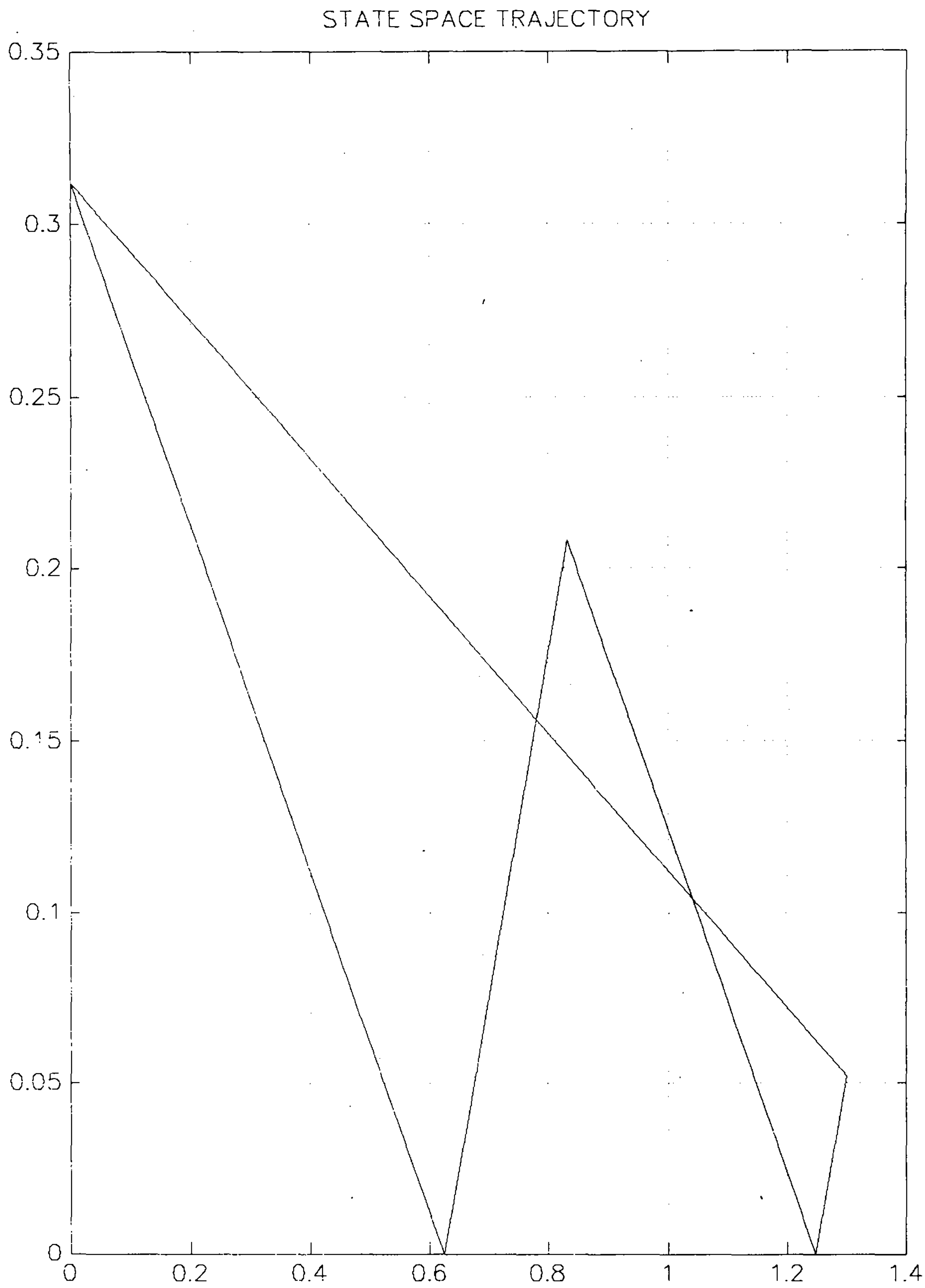

STATE SPACE TRAJECTORY
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4

FIGURE 3

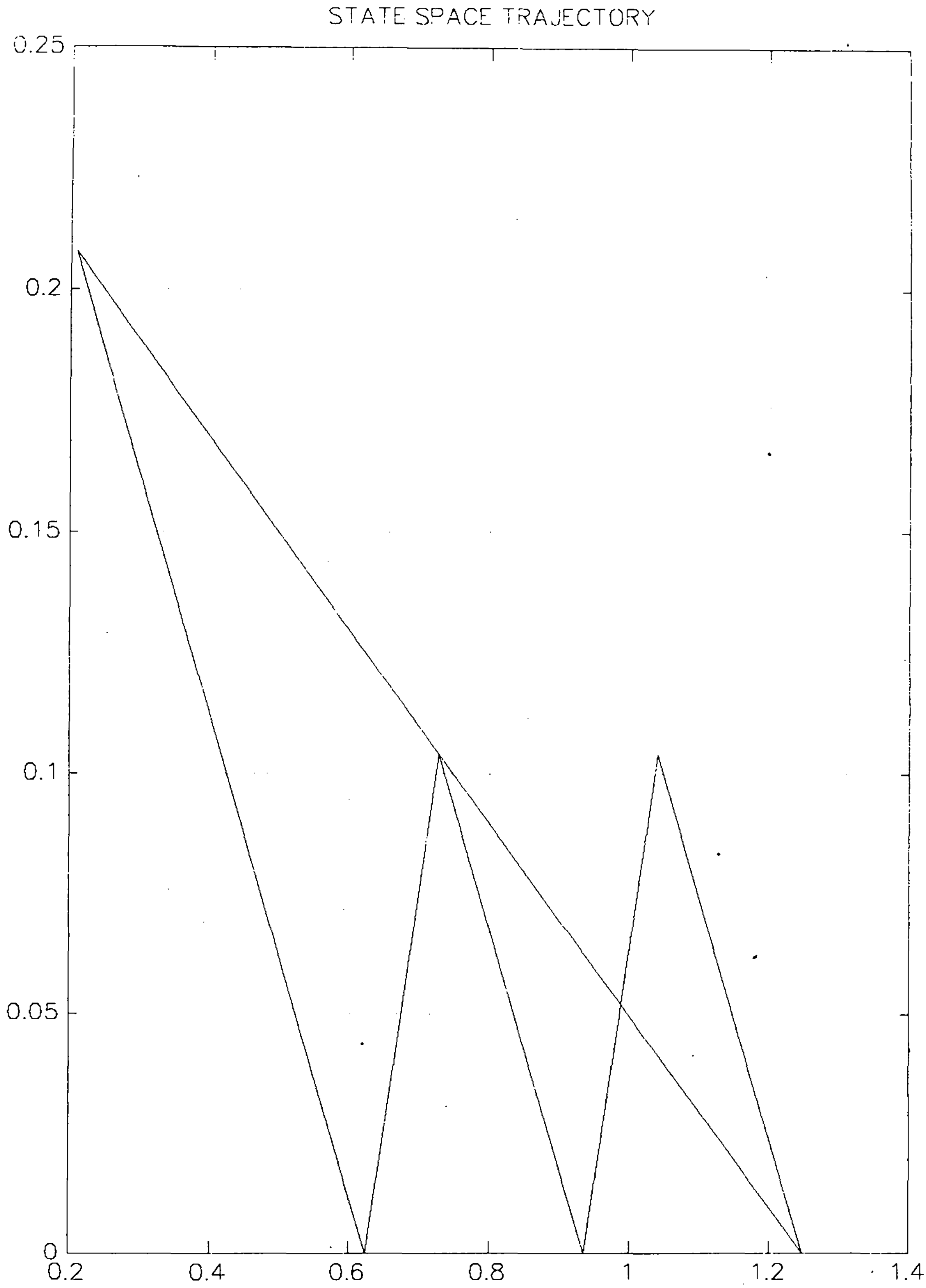



FIGURE 4

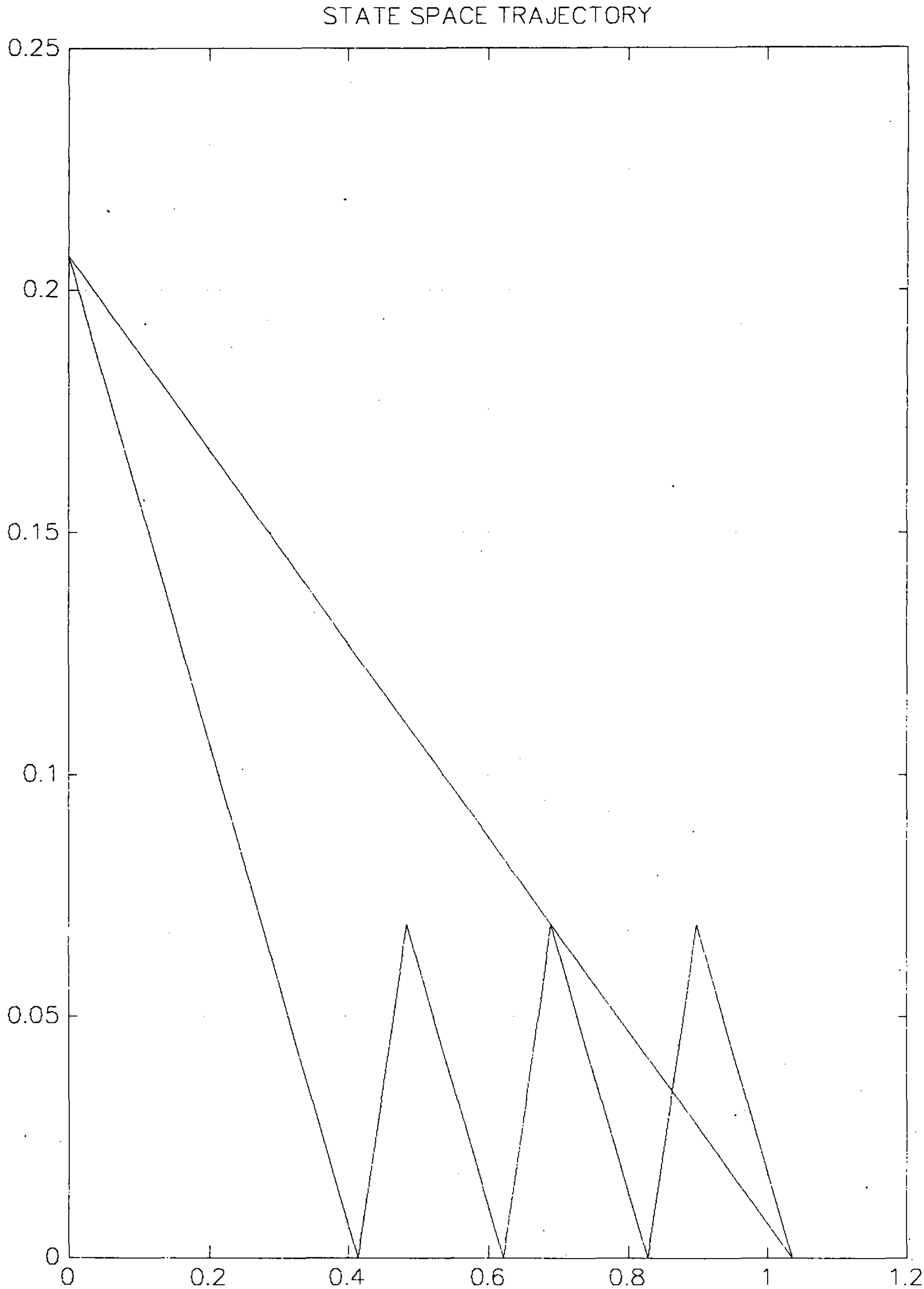



FIGURE 5

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